# Chapter 8

# ROTATIONAL MOTION -- PART I

## A.) Preliminary Comments and Basic Definitions:

**1.)** We are about to draw an almost perfect parallel between *translational* and *rotational motion*. That is, every translational concept so far covered (i.e., kinematics, Newton's Laws, energy considerations, momentum, etc.) has its rotational counterpart. The next two chapters will focus on those parallels and the general analysis of rotational systems.

**2.)** The *radian* angular measure:

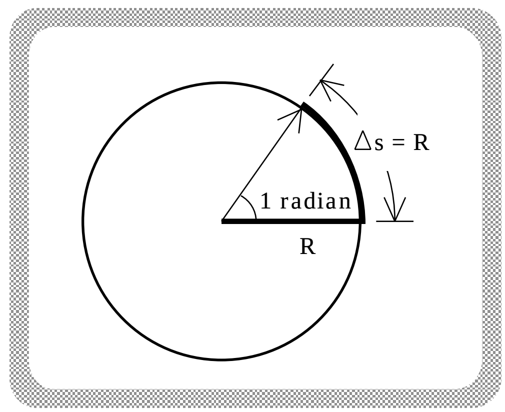



FIGURE 8.1a

**a.)** Consider a circle. If we take its radius *R* and lay it onto the *circumference of the circle*, we will create an angle whose *arc length* is equal to *R* (Figure 8.1a). Any angle that satisfies this criterion is said to have an angular measure of *one radian.*

Put another way, a *one radian angle* subtends an arc length $\Delta s$ equal to the radius of the circle (*R*).

**b.)** With this definition, a *one-half radian* angle subtends an arc length equal to *(1/2)R* (see Figure 8.1b); a *two radian* angle subtends an arc length equal to *2R* (see Figure 8.1c); and a general $\Delta\theta$ *radian angle* subtends an arc length $\Delta s$ equal to $R\Delta\theta$ (see Figure 8.1d). In other words, the most general expression relating arc-

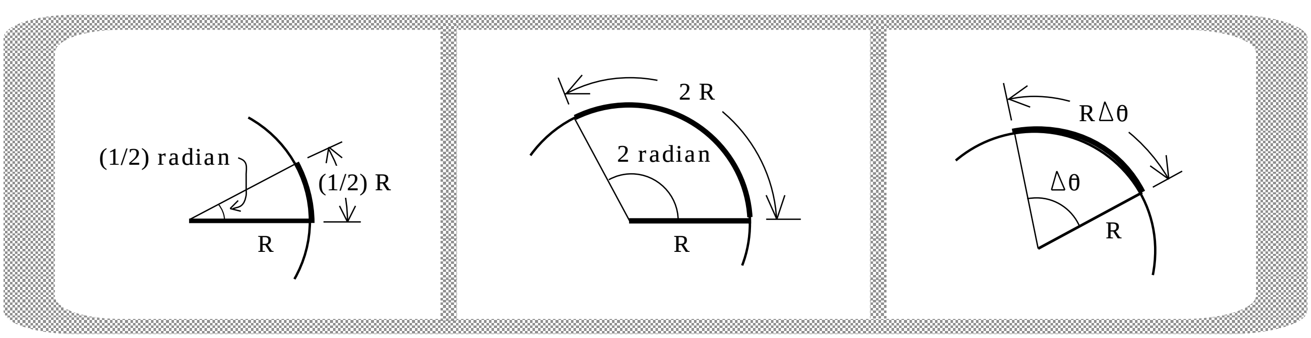



FIGURE 8.1b FIGURE 8.1c FIGURE 8.1d

length and angular measure in radians is:

$$\Delta s = R\Delta\theta .$$

**c.)** Just as *displacement* is measured in meters by coordinate variables like *x* and *y*, *angular displacement* is measured in radians by angular coordinate variables like $\theta_1$ and $\theta_2$.

**3.)** Angular Velocity:

**a.)** A point on a circle moves from a position defined by the angle $\theta_1$ to a position defined by angle $\theta_2$ in time $\Delta t$. Lines are drawn from the *center of rotation* to the point's positions at $t_1$ and $t_2$ (see Figure 8.2).

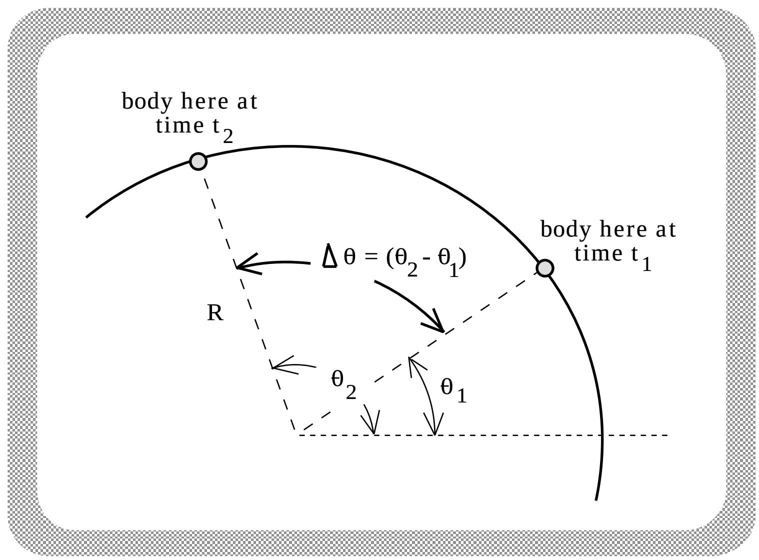



FIGURE 8.2

**b.)** The *average angular velocity* $\boldsymbol{\omega}_{avg}$ is a *vector* quantity that denotes the *angular displacement* (i.e., net change of *angular position* $\Delta\theta$ *per unit time* $\Delta t$) over some large time interval. With units of *radians/second*, it is mathematically defined as:

$$\boldsymbol{\omega}_{avg} = \Delta\boldsymbol{\theta}/\Delta t.$$

**c.)** Most elementary rotation problems assume rotational motion in the *x-y plane*. Such motion is *one dimensional* (the body isn't rotating simultaneously around *two* axes, just one--see the BIG NOTE below). As such, we can ignore the vector symbolism and write the *average angular velocity* as:

$$\omega_{avg} = \Delta\theta/\Delta t ,$$

where $\Delta\theta$ is the net *angular displacement* of the object and $\Delta t$ is the time interval over which the motion occurs.

While this appears to be a scalar equation, it is not. It matters whether the body is rotating clockwise or counterclockwise. We will account for rotational direction shortly (see BIG NOTE below).

**d.)** **Big Note** and preamble to *direction of rotation* discussion:

**i.)** A *TRANSLATIONAL velocity* vector is designed to give a reader three things: the *magnitude* of the velocity (i.e., the number of *meters per second* at which the object is moving); the *axis* or combination of axes along which the motion proceeds (unit vectors do this); and the *positive* or *negative* sense of the *direction* along those axes.

Example: A velocity vector $\mathbf{v} = -3\mathbf{i}$ tells us the object in question is traveling at 3 m/s along the *x axis* in the negative direction.

**ii.)** A *ROTATIONAL velocity* vector is also designed to give the reader three things: the *magnitude* of the rotational velocity (i.e., the number of *radians per second* through which the body moves); the *plane* in which the rotation occurs (i.e., does the rotation occur in the *x-y plane* or the *x-z plane* or some combination thereof); and the *directional sense* of the rotation (i.e., is the body rotating *clockwise* or *counterclockwise?*).

**iii.)** <u>Bottom line</u>: The *notation* used to define the *rotational velocity* vector needs to convey different information than does the *notation* used to define a *translational velocity* vector. The format used to convey the rotational information required is outlined below.

**e.)** Rotational direction:

**i.)** Consider a disk rotating in the *x-y plane* (this is the plane in which almost all of your future problems will be set). The magnitude of its angular velocity is, say, a constant $\omega = 5$ *radians/second*. Notice that although the *instantaneous, translational* direction-of-motion of each piece of the disk is constantly changing as the disk rotates, the <u>*axis*</u> about which the disk rotates always stays oriented in the same direction.

**ii.)** The <u>DIRECTION</u> of an *angular velocity vector* is defined as the direction of the <u>*axis*</u> about which the rotation occurs.

**iii.)** We have already decided that the direction about which our example's rotation occurs is along the *z axis*; the *angular velocity* vector for the problem is, therefore:

$$\boldsymbol{\omega} = (5 \text{ radians/second})\mathbf{k},$$

where ***k*** is the unit vector in the *z direction*.

**iv.)** We have just developed a clever way to mathematically convey the fact that a rotation is in the *x-y plane*. We have done so by attaching to the *angular velocity magnitude* a unit vector that defines *the axis about which the motion occurs*.

Put in a little different context, we have earmarked the *plane of rotation* by defining the direction *perpendicular to that plane* (the *z-direction* is perpendicular to the *x-y plane*).

**v.)** We still have not designated a way to define the *sense of the motion* (i.e., whether the rotation is clockwise or counterclockwise). Assuming we are looking at motion in the *x-y plane*, these two possibilities are covered nicely by assigning a positive or negative sign to the ***k*** *axis* unit vector being used to define the axis of rotation. That is:

BY DEFINITION, CLOCKWISE ROTATIONS IN THE *x-y PLANE* ARE DEFINED AS HAVING UNIT VECTOR DIRECTIONS OF -***k***, WHEREAS COUNTERCLOCKWISE ROTATIONS ARE DEFINED AS HAVING UNIT VECTOR DIRECTIONS OF +***k*** (see Figures 8.3a and 8.3b for a summary of this information).

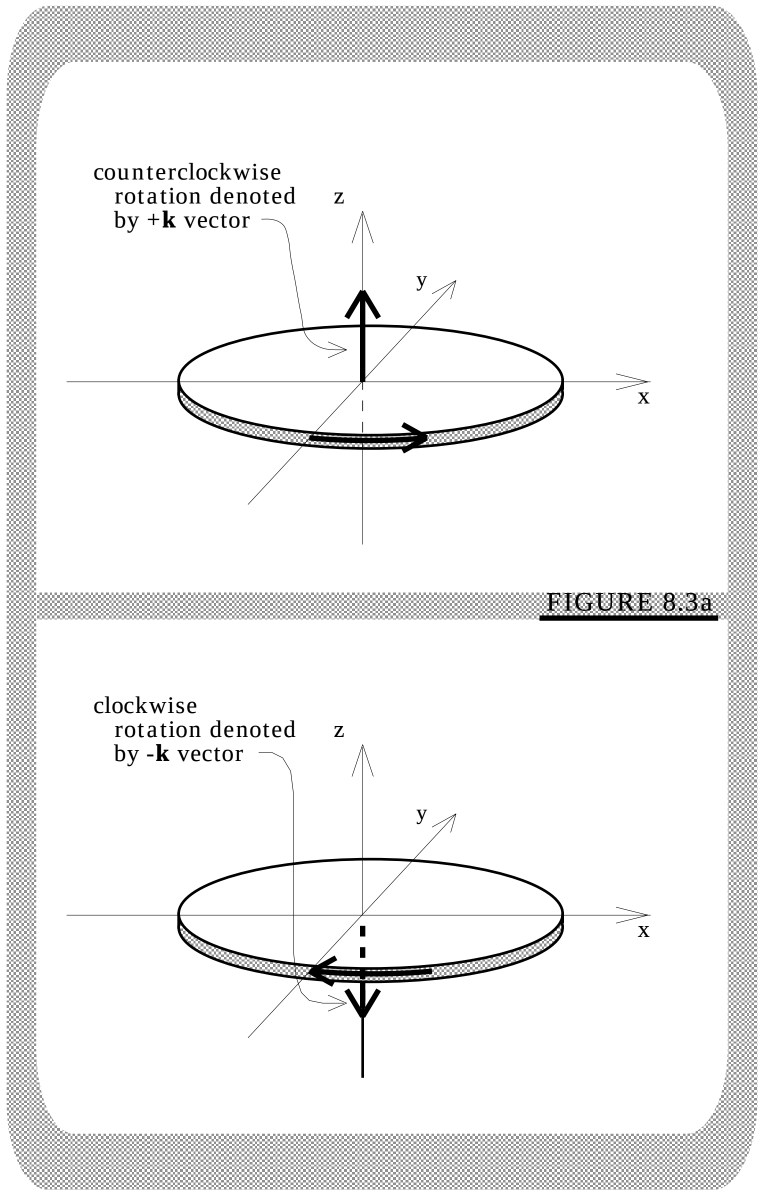


FIGURE 8.3a

FIGURE 8.3b

**Note:** This formalism is not as off-the-wall as it probably seems. Rotate a screw *counterclockwise* and it will proceed upward out of the plane in which it is embedded (that is how you unscrew a screw). Define that plane with a standard, right-handed, *x-y axis* and the screw is found to unscrew in the ***+k** direction*. A screw rotated *clockwise* will proceed into the plane in the ***-k** direction*.

As long as we always use a right-handed coordinate system (the standard within mathematics these days), the notation works nicely.

**vi.)** Mathematicians have created a mental tool by which one can remember this rotational formalism. Called *the right-hand rule*, it follows below:

Mentally place your *right hand* on the rotating disk so that when you curl your fingers, they follow the direction of the disk's rotation. Once in the correct position, extend your thumb perpendicularly out away from your fingers (i.e., in a "hitchhiker's" position). If the thumb points *upward*, the direction of the *angular velocity* is in the ***+k** direction*. If you have to flip your hand over to execute the curl, your thumb will point downward *into* the plane and the direction of *angular velocity* will be in the ***-k** direction*.

**vii.)** In summary, if our disk were rotating at 5 radians per second in the *clockwise* direction in the *x-y plane*, the *angular velocity vector* would be:

$$\boldsymbol{\omega} = (5 \text{ radians/second})(-\mathbf{k}),$$

which, for simplicity, would probably be written as:

$$\boldsymbol{\omega} = -5 \text{ rad/sec } \mathbf{k}.$$

**Note:** As all our problems will be one-dimensional (i.e., rotation in the *x-y plane*), there is no need to include the ***k*** part of this representation when solving problems. IT IS IMPORTANT TO KEEP TRACK OF THE SIGN, THOUGH. As such, this *angular velocity* vector would normally be written as $\omega = -5 \text{ rad/sec}$.

**4.)** *Instantaneous angular velocity* $\omega$ for one-dimensional motion is defined as a measure of an object's *displacement per unit time* (i.e., its rate of angular travel), measured at a particular point in time.

**a)** Mathematically, *instantaneous angular velocity* (referred to as *angular velocity* from here on) is defined as:

$$\omega = \lim_{\Delta\theta \to 0} \left( \Delta\theta / \Delta t \right)$$
$$= d\theta / dt .$$

**b.)** Example: If $\theta = at^3 - 4bt$, then $\omega = d(at^3 - 4bt)/dt = 3at^2 - 4b$.

**5.)** *Average angular acceleration* $\alpha_{avg}$ for one-dimensional rotation is defined as the rate at which the angular velocity changes *per unit time*. It is mathematically defined as:

$$\alpha_{avg} = \Delta \omega / \Delta t.$$

Its units are "radians-per-second-per-second."

**Note:** Again, this is a vector quantity even though we have not included its associated unit vector.

**a.)** Example: A man has an *angular velocity* $\omega_1 = (3\ rad/sec)$ when at $\theta_1$. Three seconds later, he is at $\theta_2$ moving with an *angular velocity* $\omega_2 = 9\ rad/sec$ (see Figure 8.4). What is his *average angular acceleration*?

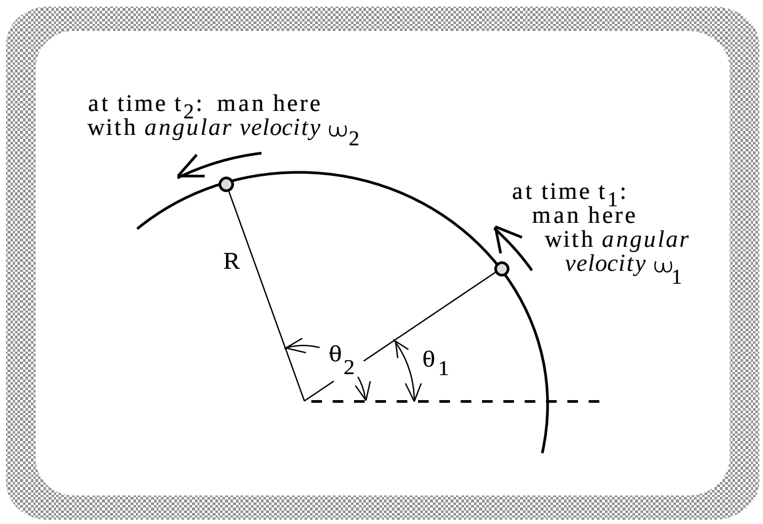



FIGURE 8.4

Solution:

$$a_{avg} = \Delta \omega / \Delta t$$
$$= (\omega_f - \omega_i)/(\Delta t)$$
$$= (9\text{ rad/s} - 3\text{ rad/s})/(3\text{ sec})$$
$$= 2\text{ rad/s}^2.$$

**6.)** *Instantaneous angular acceleration* ($\alpha$): a measure of an object's *change-of-angular-velocity per unit time*, evaluated at a particular point in time.

**a.)** Mathematically, instantaneous angular acceleration (referred to as *angular acceleration* from here on) is defined as:

$$\alpha = \lim_{\Delta\theta \to 0} \left( \Delta\omega / \Delta t \right) = d\omega / dt .$$

**b.)** Example: If $\omega = 3bt^2 - 4c$, then $\alpha = d(3bt^2 - 4c)/dt = 6bt$.

**7.)** NOTICE: For every translational parameter, we have identified a comparable rotational parameter. *Translational position* is defined using coordinates like $x$ and $y$; *angular position* is defined using angular coordinates ($\theta$*'s* measured in radians). *Velocity* is defined as $dx/dt$ in meters/second; *angular velocity* is defined as $d\theta/dt$ in radians/second. *Acceleration* is defined as $dv/dt$ in meters/second$^2$; *angular acceleration* is defined as $d\omega/dt$ in radians/second$^2$.

**8.)** Relationship between Angular Motion and Translational Motion:

**a.)** Consider a point moving with a constant *angular velocity* $\omega$ in a circular path of radius $R$. At time $t_1$, the point's *angular position* is defined by the angle $\theta_1$. At time $t_2$, its angular position is $\theta_2$ (see Figure 8.5a).

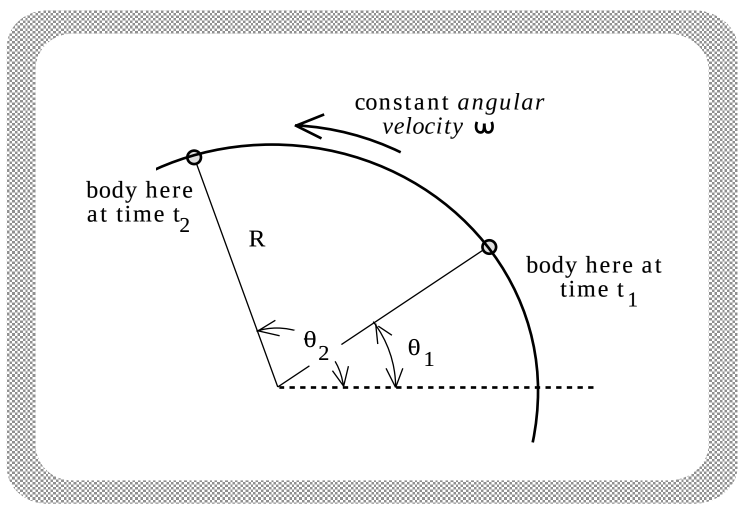



FIGURE 8.5a

**b.)** During the interval $\Delta t$, the point travels a translational distance equal to the arc length $\Delta s$ of the subtended angle $\Delta\theta = (\theta_2 - \theta_1)$ (see Figure 8.5b on next page). We know from the definition of radian measure that that arc length is:

$$\Delta s = R\Delta\theta .$$

**c.)** Dividing both sides by the time of travel yields:

$\Delta s/\Delta t = R\,(\Delta\theta/\Delta t)$.

**i.)** The left-hand side of this relationship is simply the *magnitude* of the *instantaneous translational velocity v* of the point as it moves along the arc (it is actually the magnitude of the *average* translational velocity, but because the point is moving with a constant angular velocity, the *average magnitude* and the *instantaneous magnitude* will be the same).

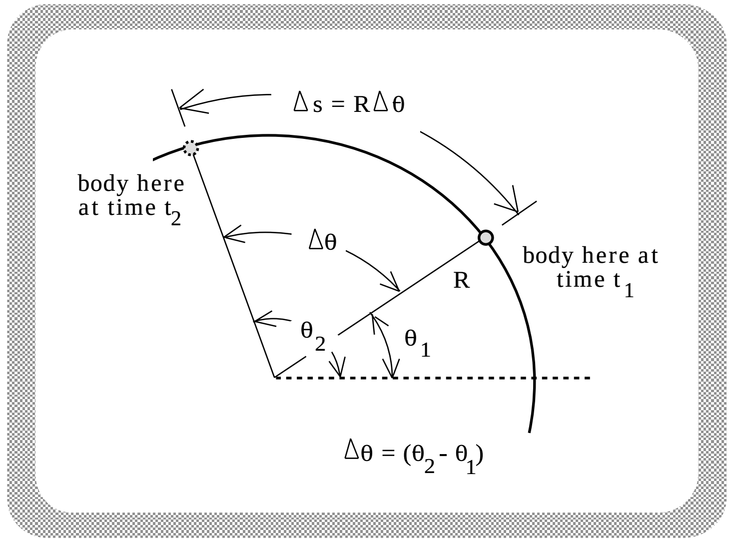



FIGURE 8.5b

**ii.)** The right-hand side of the equation is the *radius of motion* times the *magnitude* of the *instantaneous angular velocity* ($\omega$).

**iii.)** In other words, at a given instant the *magnitude* of a rotating body's *instantaneous translational velocity* at a given point will equal the *radius r of the motion* times the *magnitude* of the body's *instantaneous angular velocity* at that same instant. Mathematically, this is written:

$$v = r\omega.$$

**d.)** Through similar reasoning, the relationship between the *magnitude* of a point's *instantaneous translational acceleration* and the *magnitude* of its *instantaneous angular acceleration* at the same moment is:

$$a = r\alpha.$$

## B.) Rotational Kinematic Equations:

**1.)** Parallels:

**a.)** An object moving under the influence of a *constant acceleration* has a *velocity versus time* graph that looks like the one shown in Figure 8.6. In examining that graph at the beginning of the year, we noticed a number of useful things.

**i.)** The slope of the graph $\Delta v/\Delta t$ equals the acceleration of the object.

**ii.)** The *distance* traveled $x_2 - x_1$ between times $t_1$ and $t_2$ is equal to the *area under the graph.*

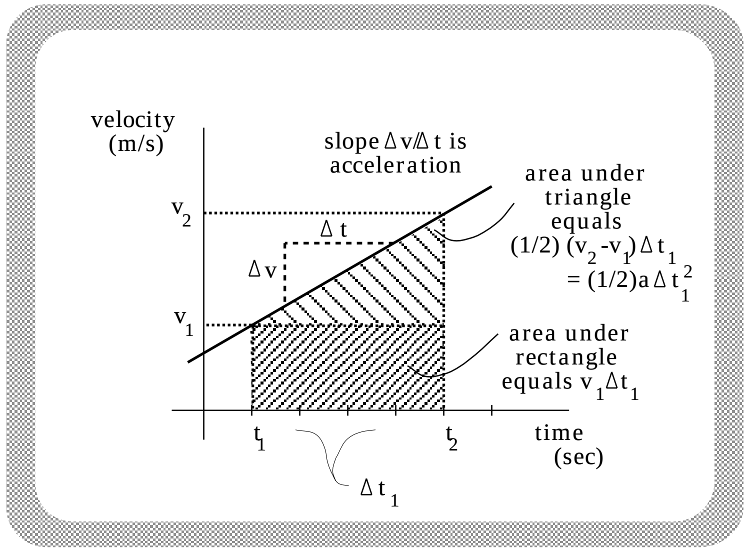



FIGURE 8.6

In fact, our beginning-of-the-year endeavors noted that the area bounded by the bottom rectangle was $v_1\Delta t_1$, the area bounded by the triangle was $(1/2)a(\Delta t_1)^2$, and that the distance traveled was $(x_2 - x_1) = v_1\Delta t_1 + (1/2)a(\Delta t_1)^2$.

**b.)** Consider now a body moving in rotational motion with *constant angular acceleration* (see Figure 8.7).

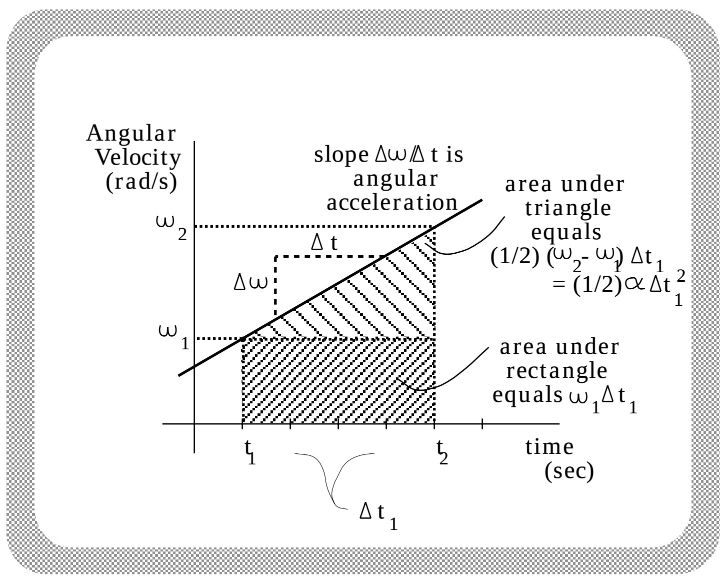



FIGURE 8.7

**i.)** Note that the <u>angular</u> *velocity versus time* graph for that motion looks exactly like its translational counterpart shown in Figure 8.6. It follows logically that:

**ii.)** The slope of that graph $\Delta\omega/\Delta t$ yields the *angular acceleration* of the motion;

**iii.)** The area under the graph yields the angular distance traveled $\theta_2 - \theta_1$.

**iv.)** In fact, with the exception of the fact that the *v's* are replaced by *ω's* and the *a's* replaced by *α's*, everything about Figure 8.6 holds for Figure 8.7.

**c.)** Having observed the parallel between the two systems of motion, it makes sense that the same reasoning that led to the translational equation $(x_2 - x_1) = v_1 \Delta t_1 + (1/2)a(\Delta t_1)^2$ will lead to its rotational image--a *rotational kinematic equation* that is exactly like its translational counterpart but with rotational parameters in place of the translational parameters. In fact, we could go through all the derivations we experienced in deriving the original set of kinematic equations and end up with all of their rotational doubles.

**d.)** This would be a waste of time. Instead, the general rotational kinematic equations are presented below next to their general translational counterparts.

$$(x_2 - x_1) = v_1 \Delta t + (1/2)a(\Delta t)^2 \quad \Rightarrow \quad (\theta_2 - \theta_1) = \omega_1 \Delta t + (1/2)\alpha(\Delta t)^2.$$

$$(x_2 - x_1) = v_{avg} \Delta t \quad \Rightarrow \quad (\theta_2 - \theta_1) = \omega_{avg} \Delta t.$$

$$v_{avg} = (v_2 + v_1) / 2 \quad \Rightarrow \quad \omega_{avg} = (\omega_2 + \omega_1) / 2.$$

$$a = (v_2 - v_1) / \Delta t \quad \Rightarrow \quad \alpha = (\omega_2 - \omega_1) / \Delta t.$$

$$(v_2)^2 = (v_1)^2 + 2a(x_2 - x_1) \quad \Rightarrow \quad (\omega_2)^2 = (\omega_1)^2 + 2\alpha(\theta_2 - \theta_1).$$

**2.)** Some Examples:

**a.)** Example 1: A turntable whose *angular velocity* is 20 rad/s *angularly accelerates* at 5 rad/s$^2$ for three seconds. What is its *angular velocity* at the end of that time period?

Solution: We know the initial *angular velocity*, the constant *angular acceleration*, and the *time interval* over which the acceleration occurs. With that we can write:

$$\alpha = (\omega_2 - \omega_1)/\Delta t$$

or $(5 \text{ rad/s}^2) = [\omega_2 - (20 \text{ rad/s})]/(3 \text{ sec})$

$$\Rightarrow \quad \omega_2 = 35 \text{ rad/s}.$$

**b.)** Example 2: What is the *average angular velocity* of the turntable mentioned in Example 1 above as it *angularly accelerates* from 20 rad/s to 65 rad/s in three seconds?

Solution: We know the *angular velocities* at the beginning and end of the time interval, and we know the angular acceleration is constant:

$$\begin{aligned} \omega_{avg} &= (\omega_2 + \omega_1)/2 \\ &= (65 \text{ rad/s} + 20 \text{ rad/s})/2 \\ &= 42.5 \text{ rad/s}. \end{aligned}$$

**c.)** Example 3: Our turntable is found to be at $\theta_2 = -26$ *radians* after having moved with an *average angular velocity* of -7 rad/s for five seconds. Where was the turntable at $t = 0$?

Solution: We know the *final* coordinate $\theta_2$, the average angular velocity $\omega_{avg}$, and the time of travel $\Delta t$:

$$\begin{aligned} (\theta_2 - \theta_1) &= \omega_{avg}\, \Delta t \\ ((-26 \text{ rad}) - \theta_1) &= (-7 \text{ rad/s})(5 \text{ sec} - 0 \text{ sec}) \\ \Rightarrow \quad \theta_1 &= +9 \text{ radians}. \end{aligned}$$

**Note:** Rotational motion treats signs just as translational motion does. Keep track of them in the equation and the equations will do everything needed to solve the problem.

**d.)** Example 4: A turntable capable of *angularly accelerating* at 12 rad/s$^2$ needs to be given an *initial angular velocity* if it is to rotate through a net 400 radians in 6 seconds. What must its initial *angular velocity* be?

Solution: We know the *angular acceleration*, the *angular distance traveled* $(\theta_2 - \theta_1)$, and the *time* of travel. To determine the initial *angular velocity* $\omega_1$:

$$\begin{aligned} (\theta_2 - \theta_1) &= \omega_1 \Delta t + (1/2)\alpha(\Delta t)^2 \\ (400 \text{ rad} - 0) &= \omega_1(6 \text{ sec}) + (1/2)(12 \text{ rad/s}^2)(6 \text{ sec})^2 \\ \Rightarrow \quad \omega_1 &= 30.7 \text{ rad/s}. \end{aligned}$$

**e.)** Example 5: A turntable *angularly accelerates* from rest to 110 rad/s in 350 radians. What is its *angular acceleration*?

Solution: We know the *initial* and *final angular velocities* and the *angular distance traveled* ($\theta_2 - \theta_1$). To get the *angular acceleration*, we use:

$$(\omega_2)^2 = (\omega_1)^2 + 2\alpha(\theta_2 - \theta_1)$$
$$\Rightarrow \quad \alpha = [(\omega_2)^2 - (\omega_1)^2]/[2(\theta_2 - \theta_1)]$$
$$= [(110 \text{ rad/s})^2 - (0)^2]/[2(350 \text{ rad} - 0)]$$
$$= 17.29 \text{ rad/s}^2.$$

## C.) A Plug for Rotational Parameters:

**1.)** Why rotational parameters? Why hassle with an "entirely new parallel system" when the old translational systems seem to do the job just fine? The answer is, "Simplicity!"

**2.)** Consider a rotating disk. Every point on the disk moves with some *translational velocity*. But as anyone who has ever played "crack the whip" knows, the farther out from the axis of rotation, the greater the translational velocity. Remember, $v_p = R_p \omega$.

**3.)** What is true but is not so obvious is that although the *translational velocity* of various pieces of the disk will differ, the angular velocity of each piece will be the same NO MATTER WHICH AXIS YOU CHOOSE TO MEASURE THAT ANGULAR VELOCITY ABOUT.

Confused? Consider the following two scenarios:

**a.)** You are sitting in a chair attached to the center of a disk. The chair is constrained to face in the same direction at all times (as the disk turns, the chair does not turn--you find you are always looking at the *Point X* shown in Figure 8.8a on the next page). The disk rotates at a constant rate through one complete revolution in, say, two seconds. What is the disk's *angular velocity* from the perspective of an axis through the *center of mass* (i.e., from where you are sitting)?

**b.)** The *angular velocity* will equal the *number of RADIANS through which the disk travels PER UNIT TIME*. As seen by you, one revolution is equal to $2\pi$ radians and the angular velocity is:

$$\omega_{\text{about cm}} = (2\pi)/(2 \text{ sec}) = \pi \text{ rad/sec}.$$

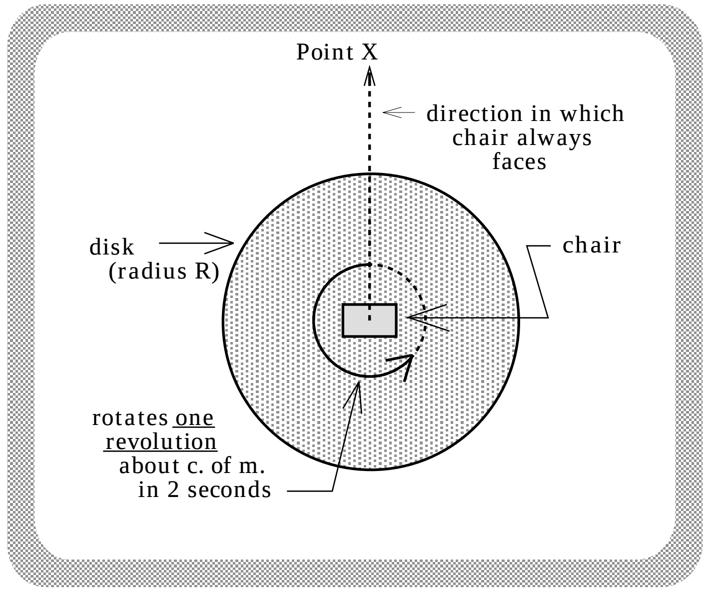



FIGURE 8.8a

**c.)** Your friend has a similar chair situated on the disk's perimeter (*Point P* in Figure 8.8b). While you are experiencing the rotation of the disk, he is experiencing the same rotation, with one big exception. Being completely self-involved, he assumes that all things revolve around him. So as the disk moves, *he sees its center rotating about himself* and not vice versa. The sketch in Figure 8.8b shows the situation.

From this perspective, how does the disk seem to rotate? It seems to make one complete revolution ($2\pi$ radians) around *Point P* in 2 seconds. That means the disk's *angular velocity* <u>*about an axis through a point on the perimeter*</u> equals:

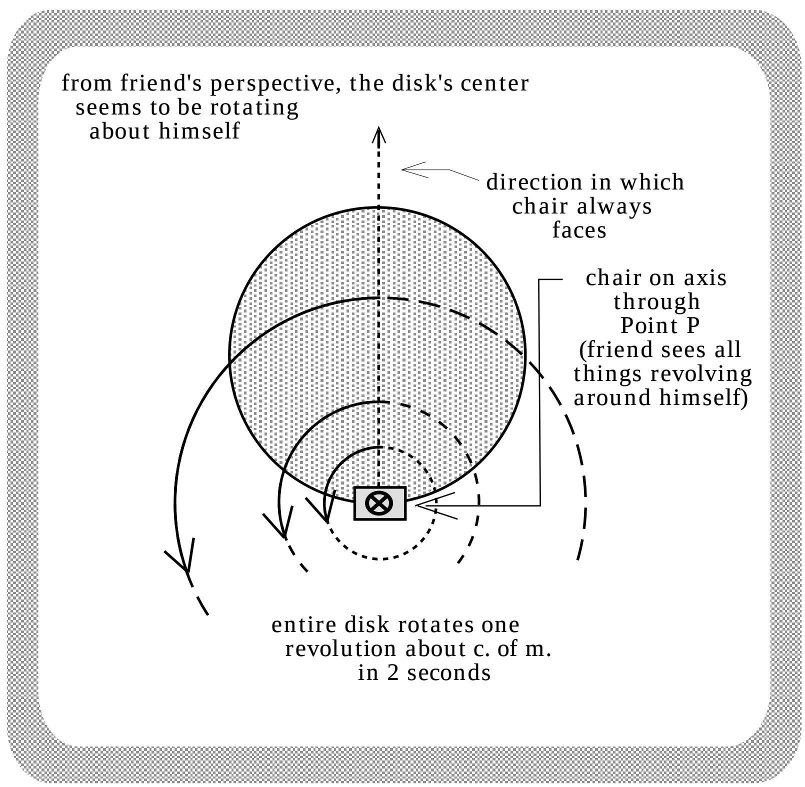



FIGURE 8.8b

$$\omega_{\text{pt.P}} = (2\pi \text{ rad})/(2 \text{ sec}) \qquad (= \pi \text{ rad/sec}).$$

**d.)** <u>Bottom line</u>: The *angular velocity* of a rotating object is the same no matter what axis is used to reference the motion. The same is true of *angular acceleration* and *angular displacement*. If you know the *value of* or have *an expression for* a rotational variable about one axis at a given instant, you know that variable at that instant about *all* axes on the rotating body.

## D.) Rotational Inertia (Moment of Inertia):

**1.)** Massive bodies have a definite tendency to resist changes in their motion. Put a truck and a feather in space, blow hard on both, and you'll find the feather is quite responsive while the truck just sits there. Why? Because the truck has more inertia--it resists changes in its motion considerably more than does the feather. The mass of a body is a quantitative measure of a body's relative tendency to resist changes in its motion. That is, saying the body has 2 kilograms of mass means that it has twice as much inertia as does a 1 kilogram mass.

**2.)** Rotating bodies have *rotational* inertia. That is, they tend to resist changes in their rotational motion. Rotational inertia is mass related--the more the mass, the greater the rotational inertia--but it is also related to how the mass is distributed relative to the *axis of rotation*. The more the mass is spread out away from the *axis of rotation*, the more rotational inertia. We need to determine a quantitative expression for the *rotational inertia* of a massive object. We will do so by considering the *rotational kinetic energy* of a spinning disk (you will not be expected to *reproduce* the following derivation).

**3.)** Consider a disk of mass $M$ and radius $R$ rotating about its central axis with a constant *angular velocity* $w$. How much *kinetic energy* is wrapped up in the disk's motion?

**a.)** Begin by defining the $i^{th}$ bit of mass $m_i$ located a distance $r_i$ meters from the *axis of rotation* and moving with an *instantaneous translational velocity* $v_i$ (Figure 8.9 looks down on the disk from above). The kinetic energy of that bit of mass will be:

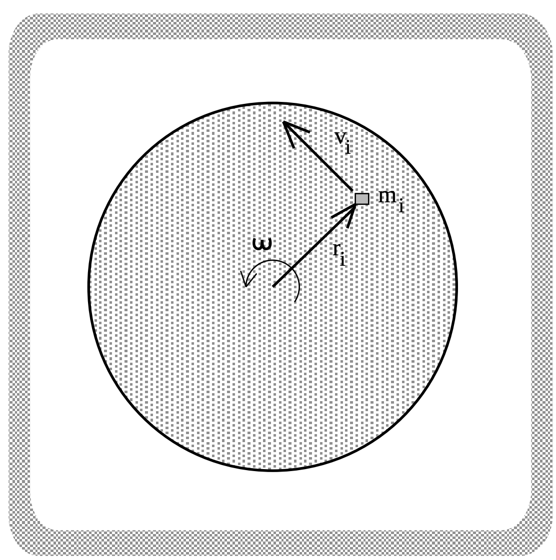



FIGURE 8.9

$$(KE)_i = (1/2)m_i v_i^2.$$

**b.)** The *total* kinetic energy of the entire disk will be the KE sum of all of the bits. That is:

$$KE = \Sigma (KE)_i = \Sigma\ [(1/2)m_i v_i^2].$$

**c.)** As we are dealing with a rotating body, it might be useful to incorporate our rotational parameters into the above expression. Remembering that the *translational velocity* $v$ of an object moving with *angular velocity* $\omega$ a distance $R$ units from the axis of rotation is $v = R\omega$, $m_i$*'s* velocity in rotational parameters must be $v_i = r_i \omega$. Substituting into the kinetic energy equation yields:

$$\begin{aligned} KE &= \Sigma\ [(1/2)m_i(r_i \omega)^2] \\ &= 1/2\ [\Sigma\ m_i r_i^2]\,\omega^2. \end{aligned}$$

**d.)** This last expression is interesting because it looks very much like a kinetic energy equation, rotational style. That is, normal kinetic energy has an equation that looks like $(1/2)m_i v_i^2$. We would expect a rotational version to have the same form: 1/2 times some *mass related term* times the *angular velocity* $\omega$ squared. That is exactly what we have above.

**e.)** Notice that the mass related term in the expression above is equal to $\Sigma\ m_i r_i^2$. This term is called *the moment of inertia* of the body (in this case, about one of its central axes). Its symbol is $I$ and its size provides us with a relative measure of the body's *rotational inertia*.

**4.)** The *moment of inertia* equation derived above is for systems of individual, discrete masses. Examples follow:

**a.)** Example 1: Consider two 3 kg masses connected by a very light (read that "massless") bar of length 4 meters (see Figure 8.10). Determine the *moment of inertia* of the system about an axis through the system's *center of mass*.

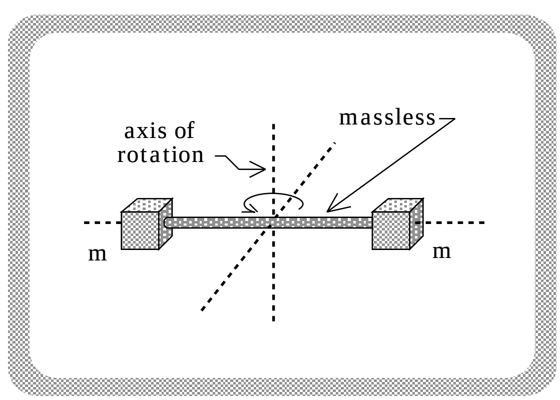



FIGURE 8.10

Solution:

$$I = \sum m_i r_i^2$$
$$= m_1 r_1^2 + m_2 r_2^2$$
$$= (3\text{ kg})(2\text{ m})^2 + (3\text{ kg})(2\text{ m})^2$$
$$= 24\text{ kg}\cdot\text{m}^2.$$

**b.)** Example 2: Using the *bar and mass* set-up presented in Example 1 above, determine the *moment of inertia* for the system about an axis through *one of the masses* (this axis is denoted in Figure 8.11).

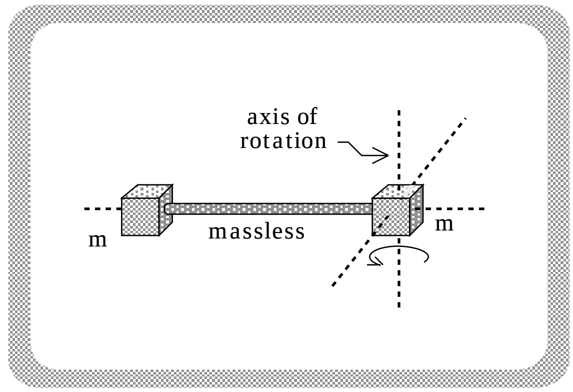



FIGURE 8.11

Solution:

$$I = \sum m_i r_i^2$$
$$= m_1 r_1^2 + m_2 r_2^2$$
$$= (3\text{ kg})(0\text{ m})^2 + (3\text{ kg})(4\text{ m})^2$$
$$= 48\text{ kg}\cdot\text{m}^2.$$

**c.)** Example 3: Consider the same set-up as above. Determine the *moment of inertia* about an axis located 2 meters to the right of the right-most mass (see Figure 8.12). Note that the massless rod would have to be extended out to the right to accommodate such a situation.

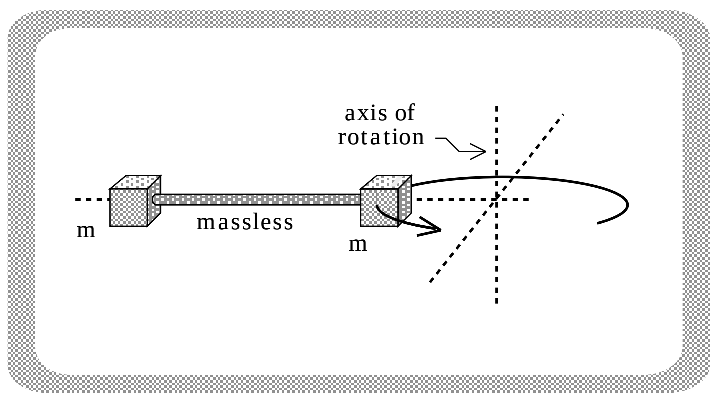



FIGURE 8.12

Solution:

$$I = \sum m_i r_i^2.$$
$$= m_1 r_1^2 + m_2 r_2^2$$
$$= (3\text{ kg})(2\text{ m})^2 + (3\text{ kg})(6\text{ m})^2$$
$$= 120\text{ kg}\cdot\text{m}^2.$$

**5.)** As the *axis of rotation* moves farther and farther from the *center of mass*, the moment of inertia increases. In fact, the moment of inertia will always be a *minimum* about an axis through the *center of mass*. There is a formula that allows one to determine the *moment of inertia* about any axis *parallel* to an axis through the *center of mass*. Called "the PARALLEL AXIS THEOREM," it states that the *moment of inertia* about any axis *P* is:

$$I_p = I_{cm} + Md^2,$$

where $I_{cm}$ is the known *moment of inertia* about a *center of mass axis* parallel to *P*, *M* is the total mass in the system, and *d* is the *distance* between the two parallel axes.

**a.)** Example 4: In Example 1 above, we calculated the *moment of inertia* about the *center of mass* of our *bar and masses* system. Using the *parallel axis theorem*, determine the moment of inertia about an axis through one of the masses (Figure 8.11).

Solution:

$$\begin{aligned} I_p &= I_{cm} + Md^2, \\ &= (24\ \text{kg}\cdot\text{m}^2) + (6\ \text{kg})(2\ \text{m})^2 \\ &= 48\ \text{kg}\cdot\text{m}^2. \end{aligned}$$

Again, this is exactly the value calculated in Example 2.

**b.)** Example 5: Determine the *moment of inertia* for our *bar and masses* system about an axis 2 meters to the right of the right-most mass (Figure 8.12):

$$\begin{aligned} I_p &= I_{cm} + Md^2, \\ &= (24\ \text{kg}\cdot\text{m}^2) + (6\ \text{kg})(4\ \text{m})^2 \\ &= 120\ \text{kg}\cdot\text{m}^2. \end{aligned}$$

This is exactly the value calculated in Example 3.

**6.)** THINK ABOUT WHAT THE MATHEMATICAL OPERATION WE HAVE BEEN EXAMINING IS ACTUALLY ASKING YOU TO DO. It says, *move out a distance "r" units from the axis of interest. If there is mass located at that distance out, multiply that mass quantity by the square of the distance "r." Do this for all possible "r" values, then sum.*

The general *moment of inertia* equation derived above works fine for systems involving groups of individual masses, but it would be cumbersome for continuous masses like the disk with which we began. To determine the *moment of inertia* for structures whose mass is extended out over a continuous volume requires Calculus. Specifically, we must solve the integral:

$$I = \int r^2 dm,$$

where *dm* is the mass found a distance *r* units from the axis of choice.

The approach for solving this integral is similar to the approach used when de-termining *center of mass* quantities for extended objects in the last chapter.

**a.)** Example: Determine the *moment of inertia* about an axis through the center of mass of a flat, homogeneous disk of mass *m* and radius *R* (assume the axis is perpendicular to the face of the disk).

**b.)** Figure 8.13 shows a hoop of differential mass *dm* with differential thickness *dr* drawn a distance *r* units from the disk's center of mass.

**c.)** We can define a *mass per unit area* function $\sigma$ in two ways: macroscopically and microscopically.

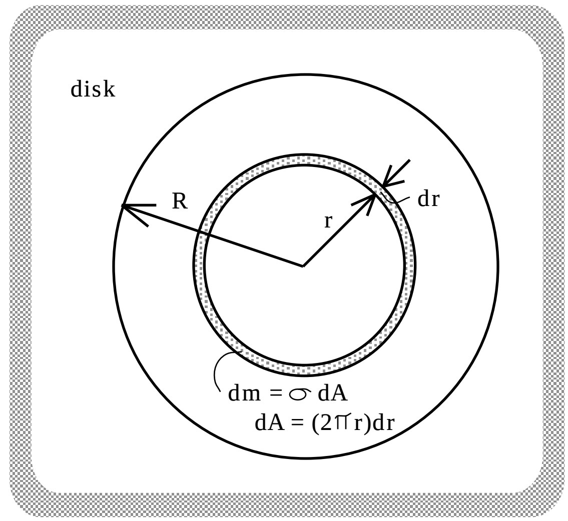


FIGURE 8.13

**i.)** Macroscopically (i.e., as a whole):

$$\sigma = M/[(\pi R^2)].$$

**ii.)** Microscopically:

$$\sigma = dm/dA,$$

where *dm* is the differential mass within the differential hoop of differential area *dA*. With this, we can write:

$$dm = \sigma dA.$$

**iii.)** Looking at Figure 8.13, we can also write:

$$\begin{aligned} dA &= (\text{circumference of hoop})(\text{hoop thickness}) \\ &= (2\pi r) \qquad dr. \end{aligned}$$

**iv.)** That means:

$$dm = \sigma dA$$
$$= \sigma[(2\pi r)dr].$$

**d.)** We are now ready to evaluate the *moment of inertia* integral:

$$I = \int r^2 dm$$
$$= \int_{r=0}^{R} r^2[\sigma(2\pi r)dr].$$

Substituting $\sigma = M/[(\pi R^2)]$ and pulling out the constants yields:

$$I = \frac{2\pi M}{\pi R^2}\int_{r=0}^{R} r^3 dr$$
$$= \frac{2M}{R^2}\left[\frac{r^4}{4}\right]_{r=0}^{R}$$
$$= \frac{2M}{R^2}\left[\frac{R^4}{4} - \frac{0^4}{4}\right]$$
$$= \frac{1}{2}MR^2.$$

This is the moment of inertia of a solid disk about an axis perpendicular to the disk's face and through its center of mass.

**e.)** There is another way to do this integral. Instead of defining an *area density function* $\sigma$ related to the mass *behind* a given area on the disk's face, we can define a *volume density function* $\rho$ related to the amount of mass that is actually wrapped up in that volume.

This density function can be expressed in two ways:

**i.)** Macroscopically, the *volume density* can be written as the total mass divided by the total volume, or:

$$\rho = M/[\pi R^2 t],$$

where $M$ is the disk's total mass, $\pi R^2$ is the area of the disk's face, and $t$ is the thickness of the disk.

**ii.)** Microscopically, the *volume density* can be written as the differential mass divided by the differential volume it occupies, or:

$$\rho = dm/dV,$$

where *dm* is the differential mass wrapped up in the cylindrical shell (a hoop with depth) and *dV* is the differential volume of that shell. Taking the differential face-area $2\pi r dr$ times the hoop thickness *t*, we get, $dV = 2\pi rt(dr)$. Putting everything together:

$$\begin{aligned}\rho &= (dm)/(dV)\\ &= (dm)/[2\pi rt(dr)].\end{aligned}$$

**iii.)** Substituting in for $\rho$ (i.e., $\rho = M/\pi R^2 t$) and solving for *dm*, we get:

$$\begin{aligned}dm &= \rho\, dV \qquad \text{(Equation A)}\\ &= \rho\,[2\pi rt(dr)]\\ &= [M/(\pi R^2 t)][2\pi rt(dr)]\\ &= (2Mr/R^2)dr.\end{aligned}$$

This leaves us with the integral:

$$I = \frac{2M}{R^2}\int_{r=0}^{R} r^3 dr,$$

which is the same integral we found when we used the *surface-area* approach.

Bottom Line: When dealing with a symmetrically shaped, homogeneous structure, either a volume density or area density function will work.

**7.)** Another example: Determine the *moment of inertia* of a rod of length *L* and mass *M* about an axis through the rod's center of mass and perpendicular to the rod's length.

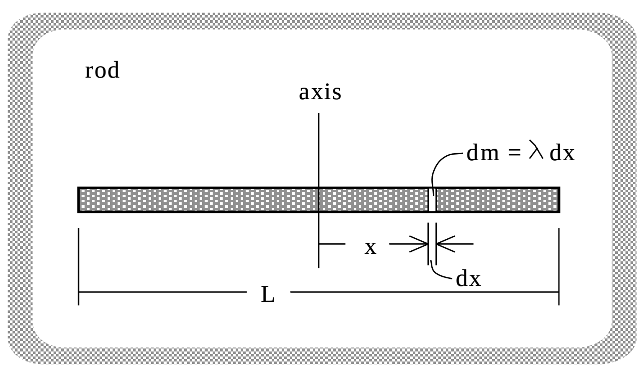


FIGURE 8.14

**a.)** Figure 8.14 shows a differential mass *dm* within a differential length *dx* located a distance *x* units from the rod's center of mass.

**b.)** Note that we could have defined either a *volume* or *area density function* for this structure. The reason we didn't? The mass is essentially distributed LINEARLY out from the axis of interest. As such, it is easier to define a *linear density function* $\lambda$ (i.e., a function that tells you the *mass* *per unit length* along the rod). From such a function, we can determine an expression for *dm*.

**c.)** As before, the linear density function $\lambda$ can be expressed in two ways: microscopically and macroscopically.

**i.)** Looking at the entire structure macroscopically, the *linear density* can be written as:

$$\lambda = M/L,$$

where *M* is the rod's total mass and *L* is its total length.

**ii.)** Looking microscopically, the *linear density* can be written as:

$$\lambda = dm/dx,$$

where *dm* is the differential mass wrapped up in the differential length *dx*. Putting everything together, we get:

$$\begin{aligned} dm &= \lambda\, dx \\ &= (M/L)dx. \end{aligned}$$

**iii.)** Letting *r* become *x*, the *moment of inertia* integral yields:

$$\begin{aligned} I &= \int r^2 dm \\ &= \int_{x=-L/2}^{L/2} x^2 \left[\frac{M}{L} dx\right] \\ &= \frac{M}{L}\left[\frac{x^3}{3}\right]_{-L/2}^{L/2} \\ &= \frac{M}{L}\left[\frac{(L/2)^3}{3} - \frac{(-L/2)^3}{3}\right] \\ &= \frac{M}{L}\left[\left(\frac{L^3}{24}\right) + \left(\frac{L^3}{24}\right)\right] \\ &= \frac{1}{12} ML^2. \end{aligned}$$

This is the *moment of inertia* about an axis perpendicular to, and through the center of mass of, a rod of mass $M$ and length $L$.

**Note:** We could have used a *volume* or *area density function* on this problem and all would have come out the same. The extra variables needed to define these functions would have canceled out in the integral, just as was the case in the earlier example.

<u>Bottom line</u>: It really doesn't matter which kind of function you use. You will end up with the same *moment of inertia* integral with each of the approaches, assuming you do each correctly.

**8.)** A list of commonly used *moment of inertia* expressions is provided in the table shown on the next page.

**Note:** Do NOT memorize the *moment of inertia* expressions found in the chart. They are provided for convenience only. If a *moment of inertia* term is required on your next test, it will either be provided or you will be asked specifically to derive it.

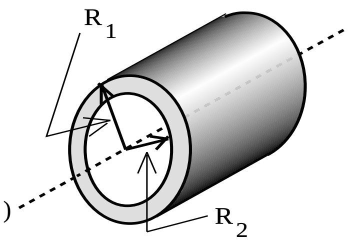
**9.) Parting shot:** General definitions, the concept of *moment of inertia*, and the *rotational kinematic equations* all have their place in your understanding of rotational motion. Nevertheless, the real players-with-power are the rotational versions of *Newton's Second Law*, *energy considerations*, and the concept of *angular momentum*. Although the chapter you are now completing is important in the sense that it presents background, the warm-up is over. The next chapter will get to the vegetables of the matter.

**NOTE:** SEE NEXT PAGE FOR *MOMENT OF INERTIA* CHART.

# MOMENT OF INERTIA EXPRESSIONS

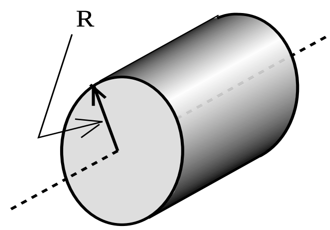
# FOR VARIOUS FORMS

Ring or Annular Cylinder about central axis



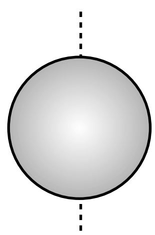

$I = (1/2)\ M(R_1^2 + R_2^2)$

Solid cylinder (or disk) about cylinder's central axis



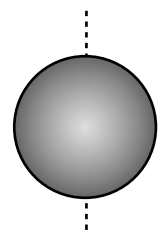

$I = (1/2)\ MR^2$

Thin spherical shell about any central axis



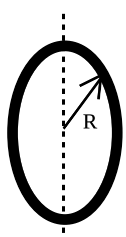
$I = (2/3)\ MR^2$

Solid sphere about any central axis



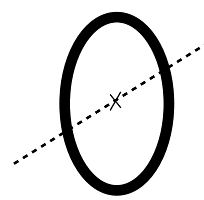
$I = (2/5)\ MR^2$

Hoop about diameter



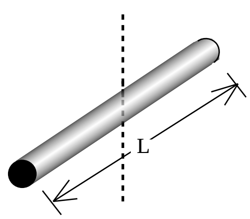

$I = (1/2)\ MR^2$

Hoop about central axis



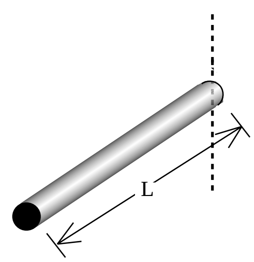
$I = MR^2$

Thin rod about axis through rod's center and perpendicular to central axis



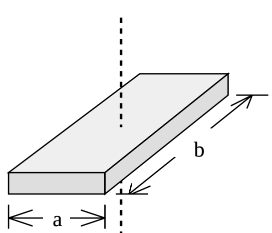

$I = (1/12)\ ML^2$

Thin rod about axis at rod's end and perpendicular to central axis



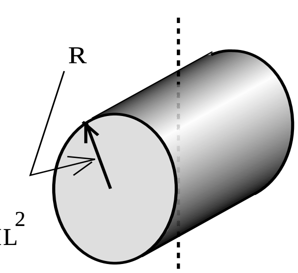

$I = (1/3)\ ML^2$

Slab about axis through center and perpendicular to slab's face




$I = (1/12)\ M(a^2 + b^2)$

Disk or Solid Cylinder about central diameter




$I = (1/4)\ MR^2 + (1/12)\ ML^2$

# QUESTIONS

**8.1) Note:** Although this question is somewhat complex, it has been included to test your ability to determine *moment of inertia* quantities for complex structures. Try it, but don't spend hordes of time on it . . . and if you get it set up but can't do the integral, don't worry about it.

A half disk has a radius $R$ and area density $ky$, where $k$ is a constant having the appropriate units and a magnitude equal to *one*. Determine the *moment of inertia* for the body:

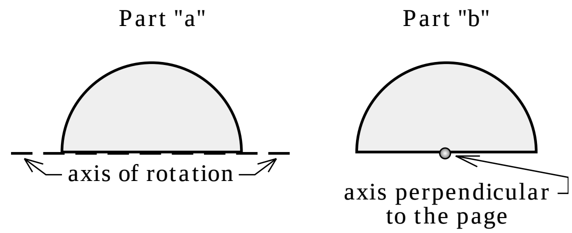



FIGURE I

**a.)** About a horizontal axis running along the bottom of the hemisphere (i.e., along the diameter), and

**b.)** About an axis perpendicular to the face and through the center of the circle defining the hemisphere's arc.

**8.2)** An automobile whose wheel radius is .3 meters moves at 54 km/hr. The car applies its brakes uniformly, slowing to 4 m/s over a 50 meter distance.

**a.)** Show that 54 km/hr is equal to 15 m/s.

**b.)** Show that a 15 m/s *car speed* corresponds to a *wheel angular velocity* of 50 radians/second and that 4 m/s corresponds to 13.33 rad/sec.

**c.)** Show that a *translational displacement* of 50 meters corresponds to a *wheel angular displacement* of 166.7 radians.

**d.)** Using rotational kinematics, determine the *angular acceleration* of one wheel.

**e.)** Using the information gleaned in *Part d*, determine the *translational acceleration* of the car.

**f.)** Knowing $\Delta\theta$, use rotational kinematics to determine the *time interval* required for the execution of the slow-down.

**g.)** Knowing $\omega_2$, use rotational kinematics to determine the *time interval* required for the slow-down. Does your solution match the one determined in *Part f*?

**h.)** Determine the *average angular velocity* of one wheel during the slow-down.

**i.)** Using rotational kinematics, determine the *angular displacement* of the wheels during the first .5 seconds of the slow-down.

**j.)** Determine *how far* the car traveled during the first .5 seconds of the slow-down.

**k.)** Without using the time interval, determine the *wheel's angular velocity* after the first .5 seconds of the slow-down.

**l.)** Determine the angular displacement of one wheel between times $t = .5$ *seconds* and $t = .7$ *seconds*.

**m.)** Once the car has slowed to 4 m/s, it begins to pick up speed. Over a 3 second period, it reaches a *wheel angular velocity* of 20 rad/sec. Using rotational kinematics, determine *how far* the car will move during that time period.

**8.3)** The earth has a *mass* of $5.98 \times 10^{24}$ kilograms, a *period* of approximately 24 hours (the *period* is the time required for one rotation about its axis), and a radius of $6.37 \times 10^{6}$ meters.

**a.)** What is the earth's *angular velocity*?

**b.)** What is the *translational velocity* of a point on the equator?

**c.)** What is the *translational velocity* of a point on the earth's surface located at an angle $60^{o}$ relative to a line from its center through its equator?

**d.)** Assuming it is a solid, homogeneous sphere, what is the earth's *moment of inertia* about its axis?